L^2-invariants

Lecture 1.

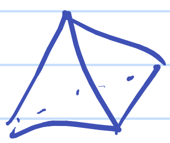
The first goal of the course is to introduce and study L^2-homology and L^2-Betti numbers.

Motivation: Euler characteristic χ.

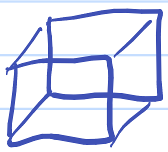
Recall the famous formula:

$$\underset{\text{vertices}}{V} - \underset{\text{edges}}{E} + \underset{\text{faces}}{F} = 2$$

holds for every platonic solid:



$4 - 6 + 4 = 2$ ✓



$8 - 12 + 6 = 2$ ✓

etc.

In fact, for every polyhedron P homeomorphic to the sphere S^2, we have

$$V - E + F = \chi(P) = \chi(S^2) = 2$$

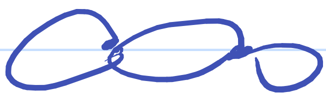
In dimension 1 the formula becomes $V - E$.

So for every polygon we have $V - E = \chi(S^1) = 0$

For a finite connected graph Γ we have

$$\pi_1(\Gamma) = F_n \ , \ n\text{-generated free group,}$$

$$\text{where } n = 1 - \chi(\Gamma).$$

Example $\quad \Gamma =$ 

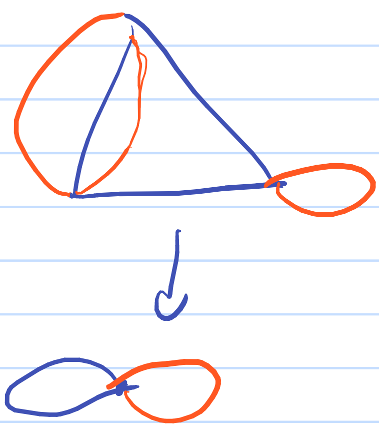
$V = 2, \ E = 4, \ \chi(\Gamma) = -2$

$\pi_1(\Gamma) \cong F_3$ ✓

Fact Euler characteristic behaves well w.r.t. coverings:

$$\chi(\Gamma) = 3 - 6 = -3$$

$3\times \qquad \times 3$



$$\Gamma \qquad \chi(\Gamma') = 1 - 2 = -1$$

$\downarrow$ 3-sheeted covering

$\Gamma' \qquad (\iff \text{index } |\Gamma' : \Gamma| = 3)$

This is in general true.

Euler characteristic is a homeomorphism invariant, whereas the numbers V, E, F are <u>not</u>!

So we replace them with Betti numbers.

<u>Def</u> X is a nice, compact topological space.

$H_i(X)$ denotes the (singular) i^{th} homology group. (an abelian group).

Let's assume that $H_i(X)$ is finitely generated.

Then $H_i(X) \cong \mathbb{Z}^{r_i} \oplus T_i$, T_i is torsion.

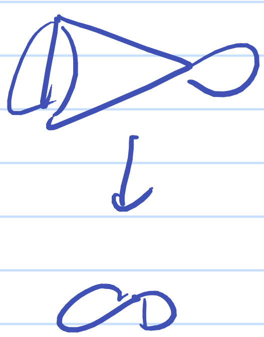
The i^{th} <u>Betti number $\beta_i(X)$</u> $= r_i = \mathrm{rk}_{\mathbb{Z}} H_i(X)$.

Euler–Poincaré formula: $\chi(X) = \sum (-1)^i \beta_i(X)$.

<u>Thm</u> If $X \to Y$ is an n-sheeted covering, then

$$\chi(Y) = n \cdot \chi(X).$$

<u>But</u>: let's go back to graphs.



Γ

$$H_i(\Gamma) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}^4 & i = 1 \\ 0 & i > 1 \end{cases}$$

Γ'

$$H_i(\Gamma') = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}^2 & i = 1 \\ 0 & i \geq 2. \end{cases}$$

Betti numbers are <u>not</u> multiplicative under finite coverings!

The point of L^2-Betti numbers $\beta_i^{(2)}(X)$ is:

- $\chi(X) = \sum (-1)^i \beta_i^{(2)}(X)$

- $X \to Y$ n-sheeted covering then
$$\beta_i^{(2)}(Y) = n \cdot \beta_i^{(2)}(X)$$

- $\beta_i^{(2)}$ is a homotopy invariant.

<u>Def</u> (CW-complexes).
A topological space X is a <u>CW-complex</u> if it comes with a filtration
$$X_0 \subseteq X_1 \subseteq \dots \quad \text{s.t.} \quad \bigcup X_n = X$$

where: X_0 is discrete. (the points in X_0 are called 0-cells)

X_{i+1} is obtained from X_i by attaching

<u>$i+1$-cells</u>, i.e. copies of the $i+1$-discs glued

to X_i via their boundaries.

X must come with the <u>weak</u> topology, i.e.

a subset A of X is open iff $A \cap X_n$

is so $\forall n \in \mathbb{N}$.

The subspaces X_n are the <u>n-skeleton</u>.

If $X = X_n \supset X_{n-1}$, then <u>$\dim X$</u> $= n$.

X is of <u>finite type</u> iff every X_n is cpct.

X is <u>finite</u> iff additionally $\dim X$ is finite

<u>Cellular homology</u>

Let X be a CW-complex, X_n being the n-skeleton.

Every X_n is obtained from X_{n-1} by gluing

n-cells $\{e_i^n \mid i \in I\}$, I some indexing set.

The gluing is a map $\partial_i^n : \partial e_i^n \longrightarrow X_{n-1}$.
$$\partial e_i^n \cong S^{n-1}$$

Now we let C_n be the free abelian group generated by $\{e_i^n \mid i \in I\}$.

C_n is the group of <u>cellular n-chains</u>

Abstractly, $C_n \cong \mathbb{Z}^I$.

[We can do the same for any ring R, obtaining the n-chains over R, isomorphic to R^I, the free R-module.]

We now define the boundary map / differential
$$\partial_n : C_n \longrightarrow C_{n-1}.$$

For $i \in I$, we look at $\partial_i^n : \partial e_i^n \longrightarrow X_{n-1}$.

We collapse X_{n-2}, obtaining $\partial_i^n : S^{n-1} \longrightarrow X_{n-1}/X_{n-2}$

Note that X_{n-1}/X_{n-2} is the wedge of $n-1$-spheres indexed by J, where $\{e_j^{n-1} \mid j \in J\}$ are the $n-1$ cells.

Now every cont. map $S^2 \longrightarrow \bigvee_J S^2$ gives degrees $(d_j)_{j \in J}$, $d_j \in \mathbb{Z}$, all but finitely many of which are zero.

We define $\partial : C_n \to C_{n-1}$ by

$$\partial(e_i^n) = \sum_{j\in J} d_j\, e_j^{n-1} \in C_{n-1}.$$

Now $C_\bullet = (C_n, \partial_n)$ is a <u>chain complex</u>,

i.e. $\partial_{n-1}\partial_n = 0 \quad \forall n$.

This is the <u>cellular chain complex</u> of X,

denoted by $C_\bullet(X) = (C_n(X), \partial_n)$.

Its homology, i.e.

$H_i(X;\mathbb{Z}) = \ker \partial_i / \operatorname{im} \partial_{i+1}$, is the <u>cellular homology</u>

of X.

<u>Metatheorem</u> If $X \simeq V$ simplicial complex, then

htpy equiv

$H_i(X;\mathbb{Z})$ agrees with simplicial and

singular homologies of V.

<u>Fact</u> cellular homology is a homotopy invariant.

Let X be a CW-complex of finite type.

Note that every $C_n(X)$ is a f.g. free abelian group. The boundary maps are matrices over $\mathbb{Z}$.

[same thing over $\mathbb{R}$]

$\therefore$ $H_i(X;\mathbb{Z})$ is a f.g. abelian group.

<u>Def</u> [Betti number]

X CW-cplx of finite type.

The i^{th} Betti number $\beta_i(X)$ is the unique integer such that

$$H_i(X;\mathbb{Z}) \cong \mathbb{Z}^{\beta_i(X)} \oplus T$$

$= \text{rank of } H_i(X;\mathbb{Z})$

$= rk(H_i(X;\mathbb{Z}))$

where T is a finite abelian group.

<u>Def</u> [Euler characteristic]

Let X be a finite CW-complex.

We define $\chi(X) = \Sigma(-1)^i c_i$ where c_i is the number of i-cells.

<u>Thm</u> [Poincaré - Euler formula].

Let X be a finite CW-cplx. Then

$$\chi(X) = \Sigma(-1)^i \beta_i(X).$$

Lemma $A \leq B$, B free abelian of finite rank.

Then $\mathrm{rk}(A/B) = \mathrm{rk}(A) - \mathrm{rk}(B)$.

Proof: Exercise (hint: Euclid's algorithm).

Proof of the theorem.

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \, \mathrm{rk}\, C_i(X)$$

Now $\mathrm{im}\, \partial_{i+1} \leq \ker \partial_i \leq C_i$

$$\mathrm{rk}\, C_i = \mathrm{rk}\,\mathrm{im}\,\partial_{i+1} + \mathrm{rk}\,\frac{\ker \partial_i}{\mathrm{im}\,\partial_{i+1}} + \mathrm{rk}\, C_i/\ker\partial_i$$

$$= \mathrm{rk}\, C_{i+1}/\ker\partial_{i+1} + \beta_i(X) + \mathrm{rk}\, C_i/\ker\partial_i$$

$$\therefore \chi(X) = (-1)^n \, \mathrm{rk}\, C_{n+1}/\ker\partial_{n+1} + \sum (-1)^i \beta_i(X)$$

$$+ \mathrm{rk}\, C_0/\ker\partial_0 .$$

Now, $C_{n+1} = 0$ and $\partial_0 = 0$, and so $\ker \partial_0 = C_0$.

Therefore $\chi(X) = \sum (-1)^i \beta_i(X)$ $\square$

How to prove the lemma in a smart way?

<u>Modules</u> R is a ring (associative, with 1).

unital
unitary

R is <u>not</u> neccesarily commutative!

<u>Def</u> A left <u>R-module</u> M is an abelian group with a left R-action, i.e. $R \longrightarrow End(M)$

<u>Example</u> Every abelian group is a $\mathbb{Z}$-module, with

$$n.a = n \cdot a .$$

We define right R-modules analogously.

<u>Def</u> Let N be a right R-mod, M a left R-mod. We define $N \otimes_R M$ to be the abelian group generated by elements $n \otimes m$, $n \in N$, $m \in M$, modulo relations:

- $(n_1 + n_2) \otimes m = n_1 \otimes m + n_2 \otimes m$
- $n \otimes (m_1 + m_2) = n \otimes m_1 + n \otimes m_2$
- $r.n \otimes m = n \otimes r.m$.

<u>Lemma</u> Let N_1, N_2 be right R-mods and

$f: N_1 \twoheadrightarrow N_2$ be a surjective R-mod map.

Then $f \otimes id_M : N_1 \otimes_R M \twoheadrightarrow N_2 \otimes_R M$ is also surjective.

<u>Proof</u> Take $n_2 \in N_2$ $m \in M$.

$\exists n_1 \in N_1 : f(n_1) = n_2$.

$\therefore f \otimes id_M (n_1 \otimes m) = n_2 \otimes m$.

But elements $n_2 \otimes m$ <u>generate</u> $N_2 \otimes M$.

<u>Def</u> M is <u>flat</u> iff the analogous statement is true for injective R-mod maps.

<u>Exercise</u> $\mathbb{Q}$ is a flat $\mathbb{Z}$-module.

<u>Proof of the Lemma.</u>

$0 \to A \to B \to A/B \to 0$ is an exact sequence of $\mathbb{Z}$-modules.

$\therefore 0 \to A \otimes \mathbb{Q} \to B \otimes \mathbb{Q} \to A/B \otimes \mathbb{Q} \to 0$

is also. But $\mathbb{Q}$ is a right-$\mathbb{Q}$-module

$\therefore$ this is a sequence of right vector spaces.

$\therefore \dim_{\mathbb{Q}} A + \dim_{\mathbb{Q}} B/A = \dim_{\mathbb{Q}} B.$

But $\dim_{\mathbb{Q}}((\mathbb{Z}^r \oplus T) \otimes \mathbb{Q}) = \dim_{\mathbb{Q}} \mathbb{Q}^r = r$
if T is torsion. $\square$

<u>Def</u> An R-mod M is <u>free</u> iff $\exists$ a set I s.t.

$$M \cong \bigoplus_I R \quad \text{as an } R\text{-mod.}$$

M is <u>projective</u> iff M is a submodule of a free module.

Alternatively, for every R-mod X, Y and R-mod maps:

$$\begin{tikzcd} & Y \arrow[d] \\ M \arrow[ur, dashed, "\exists!"] \arrow[r] & X \end{tikzcd}$$

<u>Groups rings</u>

<u>Def</u> Let R be a ring, G a group.

$$RG = \bigoplus_G R = \left\{ \sum_{g \in G} \lambda_g g \;\middle|\; \lambda_g \in R, \text{ all but finitely many are zero} \right\}$$

Multiplication is given by $\lambda g \cdot \mu h = \lambda\mu\, gh$.

Def G is a group. A CW-complex X is a <u>G-complex</u> if G acts homeomorphically on X by mapping n-cells to n-cells in such a way that if an n-cell is stabilized setwise by $g \in G$, then it is fixed pointwise by g.

A <u>G-cell</u> is a G-orbit of a cell.

If the action of G is free, X is a <u>free</u> G-complex.

<u>Key example</u> X connected CW-cpx, $G = \pi_1(X)$.

The universal covering $\tilde{X}$ is a free G-complex.

<u>Remark</u> If X is a G-complex, then $H_i(X;\mathbb{Z})$ is a $\mathbb{Z}G$-module $\forall i$.

<u>Key example again</u>

For every n-cell e_i^n in X, choose a lift $\tilde{e}_i^n$ in $\tilde{X}$. Now every other lift of e_i^n is obtained <u>uniquely</u> as $g \cdot \tilde{e}_i^n$ for some $g \in G$.

Take an n-chain $x \in C_n(\tilde{X})$.

We have $x = \sum \lambda_e e$, finite sum, where e ranges over n-cells in $\tilde{X}$. Hence

$$x = \sum \underbrace{\lambda_{g,i}\, g}_{\in \mathbb{Z}G} \cdot \tilde{e}_i^n$$

So $C_n(\tilde{X}) \cong \mathbb{Z}G^{\#\, n\text{-cells in } X}$ as a left $\mathbb{Z}G$-module.

$\therefore$ $C_n(\tilde{X})$ is a f.g. free $\mathbb{Z}G$-module!

Boundary maps can now be seen as finite matrices over $\mathbb{Z}G$.

<u>But</u>: $\tilde{X} \simeq *$ is contractible.

$$\therefore \quad H_i(\tilde{X}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{o/w} \end{cases}.$$

$\therefore$ the chain complex

$$\to C_n(\tilde{X}) \to \dots \to C_0(\tilde{X}) \xrightarrow{\varepsilon} \mathbb{Z}$$

is <u>exact</u>, where ε is the projection

$$C_0(\tilde{X}) = \ker \partial_0 \to \ker \partial_0 / \operatorname{im} \partial_1 = H_0(\tilde{X}; \mathbb{Z}) \cong \mathbb{Z}$$

[ϵ is also the augmentation map:

$$\mathbb{Z}G \to \mathbb{Z}$$
$$\sum \lambda_g g \mapsto \sum \lambda_g$$

on every coordinate].

<u>Def</u> An exact chain cplx

trivial RG-mod

$$\cdots \to C_n \to \cdots C_0 \to R$$

of RG-modules is called a <u>resolution</u>.

The resolution is free / projective iff all C_i's are.

<u>Note</u> every left RG-mod is also a <u>right</u> RG-mod via $a \cdot \lambda g = \lambda g^{-1} \cdot a$.

<u>Def</u> G a group; M an RG-mod,

$C_\bullet$ a projective resolution of the trivial RG-mod R.

The <u>group homology</u> of G with coefficients in M

is

$$H_i(G;M) = H_i(C_\bullet \otimes M).$$

<u>Key fact</u>: this is independent of $C_\bullet$.